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A note on cubic polynomial interpolation

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ABSTRACT

"The NURBS Book" [L. Piegl, W. Tiller, The NURBS Book, second edn, Springer, 1997] is very popular in the fields of computer aided geometric design (CAGD) and geometric modeling. In Section 9.5.2 of the book, the well-known problem of the local cubic spline approximation is discussed. The key in local cubic spline approximation is cubic polynomial interpolation. In this short paper, we present the concept of single-side/double-side cubic curves and obtain the necessary and sufficient condition of a cubic curve being a single-side/double-side curve. Based on this result, for some cases of two end tangents being nearly parallel we present a new method for the problem of cubic polynomial interpolation. We also point out a flaw in Section 9.5.2 of the book and give the correction result.

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Section 9.5.2 (from Page 441 to Page 453) in [1] (hereafter referred to as the "book") studies the following

Cubic Interpolation Problem (CIP): Construct a cubic polynomial interpolating given points $\mathbf{P}_0$ (the start point) and $\mathbf{P}_3$ (the end point), tangenting the given unit vectors $\mathbf{T}_0$ to $\mathbf{P}_0$ and $\mathbf{T}_3$ to $\mathbf{P}_3$, and passing through a given point $\bar{\mathbf{P}}$.

For the case $\mathbf{T}_0$, $\mathbf{T}_3$ and $\mathbf{P}_3 - \mathbf{P}_0$ are not coplanar and $\mathbf{T}_0$ and $\mathbf{T}_3$ are not nearly parallel, one can find a neat solution of the problem in [1]. But there is a flaw in the book, that is, the method in the book may have no solution when $\mathbf{T}_0$, $\mathbf{T}_3$ and $\mathbf{P}_3 - \mathbf{P}_0$ are coplanar. In addition, in the case that $\mathbf{T}_0$ and $\mathbf{T}_3$ are nearly parallel we present a new method to solve the CIP.

The following are the main steps of constructing the cubic polynomial in the case that $\mathbf{T}_0$, $\mathbf{T}_3$ and $\mathbf{P}_3 - \mathbf{P}_0$ are coplanar in the book (on page 446):

1. assign a parameter, $\bar{u}$, to the given point $\bar{\mathbf{P}}$ of CIP by accumulating chord lengths. In our case,

$$\bar{u} = \frac{\|\bar{\mathbf{P}} - \mathbf{P}_0\|}{\|\bar{\mathbf{P}} - \mathbf{P}_0\| + \|\mathbf{P}_3 - \bar{\mathbf{P}}\|}; \tag{1}$$

2. assign a tangent, $\mathbf{T}_{\bar{p}}$, at $\bar{\mathbf{P}}$ by (9.29) and (9.31) in the book (please find the detail on the pages of 384–386 in the book);
3. set up the following equations

$$\bar{\mathbf{P}} = s^3\mathbf{P}_0 + 3s^2t\mathbf{P}_1 + 3st^2\mathbf{P}_2 + t^3\mathbf{P}_3, \tag{2}$$

$$\mathbf{T}_{\bar{p}} \times (\mathbf{P}_1^2 - \mathbf{P}_0^2) = 0, \tag{3}$$

where $s = 1 - \bar{u}$, $t = \bar{u}$, and $\mathbf{P}_0^2 = s^2\mathbf{P}_0 + 2st\mathbf{P}_1 + t^2\mathbf{P}_2$, $\mathbf{P}_1^2 = s^2\mathbf{P}_1 + 2st\mathbf{P}_2 + t^2\mathbf{P}_3$. $\mathbf{P}_1^2$ and $\mathbf{P}_0^2$ are obtained from the deCasteljau algorithm and lie on the line defined by $\bar{\mathbf{P}}$ and $\mathbf{T}_{\bar{p}}$.

For the above CIP, the following result holds.

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Lemma 0.1. *If $\mathbf{P}_3 - \mathbf{P}_0$, $\mathbf{T}_0$ and $\mathbf{T}_3$ are not coplanar, the necessary and sufficient condition that CIP has a solution (i.e., there exists a parameter t, $0 < t < 1$, such that (2) holds) is that*

$$0 < M < 1, \tag{4}$$

where $M = \frac{\langle \mathbf{T}_0 \times \mathbf{T}_3, \bar{\mathbf{P}} - \mathbf{P}_0 \rangle}{\langle \mathbf{T}_0 \times \mathbf{T}_3, \mathbf{P}_3 - \mathbf{P}_0 \rangle}$.

We will prove this lemma later. If we denote $\mathbf{T}^{\perp} = \frac{\mathbf{T}_0 \times \mathbf{T}_3}{\|\mathbf{T}_0 \times \mathbf{T}_3\|}$ and

$$\mathbf{P}_3 - \mathbf{P}_0 = c_0\mathbf{T}_0 + c_3\mathbf{T}_3 + c\mathbf{T}^{\perp},$$
$$\bar{\mathbf{P}} - \mathbf{P}_0 = \bar{c}_0\mathbf{T}_0 + \bar{c}_3, \mathbf{T}_3 + \bar{c}\mathbf{T}^{\perp}$$

then

$$M = \frac{\langle \mathbf{T}_0 \times \mathbf{T}_3, \bar{\mathbf{P}} - \mathbf{P}_0 \rangle}{\langle \mathbf{T}_0 \times \mathbf{T}_3, \mathbf{P}_3 - \mathbf{P}_0 \rangle} = \frac{\bar{c}}{c}. \tag{5}$$

Eq. (5) shows that the geometric meaning of (4) is that the projections of $\bar{\mathbf{P}} - \mathbf{P}_0$ and $\mathbf{P}_3 - \mathbf{P}_0$ to $\mathbf{T}^{\perp}$ have the same direction but the projection of $\bar{\mathbf{P}} - \mathbf{P}_0$ is shorter.

The key of CIP is to find $\mathbf{P}_1$ and $\mathbf{P}_2$ such that (2) holds. But we found that in the case $\mathbf{T}_0 \parallel \mathbf{T}_3$, (2) does not always have a solution if the parameter $\bar{u}$ is priorly assigned. The following is a counter-example that (2) has no solution.

Assuming $\mathbf{P}_1 = \mathbf{P}_0 + \alpha\mathbf{T}_0$ and $\mathbf{P}_2 = \mathbf{P}_3 + \beta\mathbf{T}_3$ and substituting them into (2) yield

$$3\alpha s^2 t\mathbf{T}_0 + 3st^2\beta\mathbf{T}_3 = \bar{\mathbf{P}} - (s^3 + 3s^2t)\mathbf{P}_0 - (t^3 + 3st^2)\mathbf{P}_3. \tag{6}$$

(6) is the same as (9.107) in [1]. Let $\mathbf{P}_0 = (0, 0)^{\mathrm{T}}$, $\mathbf{P}_3 = (3, 0)^{\mathrm{T}}$, $\mathbf{T}_0 = \mathbf{T}_3 = (0, 1)^{\mathrm{T}}$ and $\bar{\mathbf{P}} = (1, \bar{y})^{\mathrm{T}}$, then (6) is equivalent to

$$\begin{cases} 1 - 3(t^3 + 3st^2) = 0, \\ 3\alpha s^2 t + 3st^2\beta = \bar{y}, \end{cases} \tag{7}$$

where $s = 1 - t$ and $t = \bar{u}$. Since the first equation of (7) does not rely on α and β, it should be an identity if (6) has a solution. However, this is impossible if the parameter $\bar{u}$ of $\bar{\mathbf{P}}$ is priorly assigned. In the current example, from (1), $\bar{u} = \frac{\sqrt{1+\bar{y}^2}}{\sqrt{1+\bar{y}^2}+\sqrt{4+\bar{y}^2}}$. Let $\bar{y} = \sqrt{3}$. Then $t = \bar{u} = \frac{2}{2+\sqrt{7}}$, $s = \frac{\sqrt{7}}{2+\sqrt{7}}$. Denoting $f(\bar{y}) = 1 - 3(t^3 + 3st^2)$, we have $f(\sqrt{3}) = 1 - \frac{3(2^3+3\cdot 2^2\sqrt{7})}{(2+\sqrt{7})^3} = \frac{26-17\sqrt{7}}{(2+\sqrt{7})^3} \neq 0$. Therefore, (6) has no solution.

From (6), it holds that

$$\bar{\mathbf{P}} - \mathbf{P}_0 = 3\alpha s^2 t\mathbf{T}_0 + 3\beta st^2\mathbf{T}_3 + (t^3 + 3st^2)(\mathbf{P}_3 - \mathbf{P}_0). \tag{8}$$

The above equation shows that $\bar{\mathbf{P}}$ has to be on the same plane, say Π, determined by $\mathbf{P}_0$, $\mathbf{P}_3$, $\mathbf{T}_0$ and $\mathbf{T}_3$. In addition, we rewrite (3) as follows ((9.108) in [1])

$$s(s - 2t)(\mathbf{T}_{\bar{p}} \times \mathbf{T}_0)\alpha + t(2s - t)(\mathbf{T}_{\bar{p}} \times \mathbf{T}_3)\beta = 2st(\mathbf{T}_{\bar{p}} \times (\mathbf{P}_0 - \mathbf{P}_3)). \tag{9}$$

Proof of Lemma 0.1. According to (8), it holds that

$$t^3 + 3st^2 = M. \tag{10}$$

Therefore, CIP has a solution is equivalent to (10) has a root in the interval $(0, 1)$. Denoting $f(t) = t^3 + 3st^2 - M = 3t^2 - 2t^3 - M$, we have $f'(t) = 6t(1 - t) > 0$ for $0 < t < 1$. Therefore, the necessary and sufficient condition of $f(t) = 0$ having a solution in the interval $(0, 1)$ is that $f(0) = -M < 0$ and $f(1) = 1 - M > 0$, i.e, (4) holds. If the condition (4) holds, we can obtain a parameter $\bar{u}$, the only root of $f(t) = 0$ in interval $(0, 1)$, for $\bar{\mathbf{P}}$. Lemma 0.1 is proved. ♣

In the following, we discuss case by case the solution of CIP in the case $\mathbf{T}_0$, $\mathbf{T}_3$ and $\mathbf{P}_3 - \mathbf{P}_0$ being coplanar.

Case 1: $\mathbf{T}_0$ and $\mathbf{T}_3$ are not parallel. This case is discussed in the book (last two lines on page 446)

Case 2: $\mathbf{T}_0 \parallel \mathbf{T}_3$. In this case, there exist four constants $a, b, \bar{a}, \bar{b}$ such that $\bar{\mathbf{P}} - \mathbf{P}_0 = \bar{a}\mathbf{T}_0 + \bar{b}\mathbf{T}_0^{\perp}$ and $\mathbf{P}_3 - \mathbf{P}_0 = a\mathbf{T}_0 + b\mathbf{T}_0^{\perp}$, where $\mathbf{T}_0^{\perp}$ is a unit vector which is perpendicular to both $\mathbf{T}_0$ and the normal vector of Π. On the other hand, $\mathbf{T}_3 = \varepsilon\mathbf{T}_0$ $(\varepsilon = \pm 1)$ because $\mathbf{T}_3 \parallel \mathbf{T}_0$ and both of them are the unit vectors. Thus, (8) can be converted to

$$\begin{cases} b(t^3 + 3st^2) = \bar{b}, \\ 3\alpha s^2 t + 3\varepsilon st^2\beta = \bar{a} - a(t^3 + 3st^2). \end{cases} \tag{11}$$

We can reasonably assume that $0 < t = \bar{u} < 1$. By this assumption, the first equation of (11) shows that b and $\bar{b}$ have the same sign. Thus, $\bar{\mathbf{P}} - \mathbf{P}_0 \parallel \mathbf{T}_0$ if and only if $\mathbf{P}_3 - \mathbf{P}_0 \parallel \mathbf{T}_0$.

Case 2.1: $\mathbf{P}_3 - \mathbf{P}_0$ is not parallel to $\mathbf{T}_0$. In this case, we can not priorly assign a parameter for $\bar{\mathbf{P}}$. We have to obtain a parameter t, $0 < t < 1$, for $\bar{\mathbf{P}}$ from the first equation of (11). Since $\mathbf{P}_3 - \mathbf{P}_0$ is not parallel to $\mathbf{T}_0$, we know that both b and $\bar{b}$ are not zeros. Similar to Lemma 0.1, in this case the CIP has a solution if and only if

$$0 < \frac{\bar{b}}{b} < 1. \tag{12}$$

Unlike assigning a tangent $\mathbf{T}_{\bar{p}}$ at $\bar{\mathbf{P}}$ in the book, we present here an optimization method to determine α and β. It can be seen that this optimization method for CIP produces fair curves. In concentrate, we determine the parameters α and β of $\mathbf{C}(t)$ by minimizing

$$\min_{\alpha,\beta} \int_0^1 \|\mathbf{C}''(t)\|^2 \mathrm{d}t \tag{13}$$

under the restriction of the second equation of (11) wherein $s = 1 - \bar{u}$ and $t = \bar{u} \in (0, 1)$, the unique root of the first equation of (11) in the interval $(0, 1)$, where

$$\mathbf{C}(t) \quad = \quad s^3\mathbf{P}_0 + 3s^2t\mathbf{P}_1 + 3st^2\mathbf{P}_2 + t^3\mathbf{P}_3, \;\; s = 1 - t, \tag{14}$$

and $\mathbf{P}_1 = \mathbf{P}_0 + \alpha\mathbf{T}_0$ and $\mathbf{P}_2 = \mathbf{P}_3 + \beta\mathbf{T}_3$.

It is well-known that $\mathbf{C}(t)$ is fairer if it satisfies (13). According to (14), it holds that

$$\begin{aligned}
&\mathbf{C}(t) = \mathbf{P}_0 + 3s^2t\alpha\mathbf{T}_0 + 3st^2\beta\mathbf{T}_3 + (t^3 + 3st^2)(\mathbf{P}_3 - \mathbf{P}_0),\\
&\mathbf{C}'(t) = 3(1-t)(1-3t)\alpha\mathbf{T}_0 + 3t(2-3t)\beta\mathbf{T}_3 + 6t(1-t)(\mathbf{P}_3 - \mathbf{P}_0),\\
&\mathbf{C}''(t) = 3(6t-4)\alpha\mathbf{T}_0 + 3(2-6t)\beta\mathbf{T}_3 + 6(1-2t)(\mathbf{P}_3 - \mathbf{P}_0),\\
&\mathbf{C}'''(t) = 18\alpha\mathbf{T}_0 - 18\beta\mathbf{T}_3 - 12(\mathbf{P}_3 - \mathbf{P}_0).
\end{aligned} \tag{15}$$

According to (15) and noting that $\mathbf{C}'''(t)$ is a constant vector, we obtain directly

$$\begin{aligned}
\int_0^1 \|\mathbf{C}''(t)\|^2 \mathrm{d}t &= \int_0^1 \langle \mathbf{C}''(t), \mathbf{C}''(t)\rangle \mathrm{d}t = \left(\langle \mathbf{C}'(t), \mathbf{C}''(t)\rangle - \langle \mathbf{C}'''(t), \mathbf{C}(t)\rangle\right)\Big|_{t=0}^{t=1}\\
&= 36(\alpha^2 - \varepsilon\alpha\beta + \beta^2 - c_0\alpha + \varepsilon c_0\beta) + 12\|\mathbf{P}_3 - \mathbf{P}_0\|^2,
\end{aligned} \tag{16}$$

where $\langle \mathbf{A}, \mathbf{B}\rangle$ is the inner product of the vectors $\mathbf{A}$ and $\mathbf{B}$, $\varepsilon = \langle \mathbf{T}_0, \mathbf{T}_3\rangle = \pm 1$ and $c_0 = \langle \mathbf{T}_0, \mathbf{P}_3 - \mathbf{P}_0\rangle$.

Denote by $k_1 = 3(1-\bar{u})^2\bar{u}$, $k_2 = 3\varepsilon(1-\bar{u})\bar{u}^2$, $k_3 = a(\bar{u}^3 + 3(1-\bar{u})\bar{u}^2) - \bar{a}$, and

$$f(\alpha, \beta) = \alpha^2 - \varepsilon\alpha\beta + \beta^2 - c_0\alpha + \varepsilon c_0\beta,$$

where $a = \langle \mathbf{T}_0, \mathbf{P}_3 - \mathbf{P}_0\rangle$, $\bar{a} = \langle \mathbf{T}_0, \bar{\mathbf{P}} - \mathbf{P}_0\rangle$ and $\bar{u} \in (0, 1)$ is the solution of the first equation of (11). Then, the optimization problem of (13) is equivalent to the following problem:

$$\min_{\alpha,\beta}\{f(\alpha, \beta);\, k_1\alpha + k_2\beta + k_3 = 0\}. \tag{17}$$

By Lagrange multiplier, (17) has the following solution:

$$\alpha = \frac{1}{3}(c_0 + \lambda(2k_1 + \varepsilon k_2)), \qquad \beta = \frac{1}{3}(-\varepsilon c_0 + \lambda(\varepsilon k_1 + 2k_2)), \qquad \lambda = -\frac{k_1c_0 - k_2\varepsilon c_0 + 3k_3}{2(k_1^2 + \varepsilon k_1k_2 + k_2^2)}. \tag{18}$$

Our later examples show that α and β determined by (18) produce fair curves.

Case 2.2: $\mathbf{P}_3 - \mathbf{P}_0 \parallel \mathbf{T}_0$. We rule out this straight line case as in the book.

For discussing the third case, we introduce the following concept for CIP. we assume $\mathbf{C}(t) = \sum_{i=0}^{3}\mathbf{P}_iB_{i,3}(t)$ the cubic curve obtained by the CIP and denote $\mathbf{C}_\pi(t)$ the orthogonal projection of $\mathbf{C}(t)$ onto the plane π, where π is the plane determined by points $\mathbf{P}_0, \mathbf{P}_3, \bar{\mathbf{P}}$ (we can reasonably assume that $\mathbf{P}_0, \mathbf{P}_3, \bar{\mathbf{P}}$ are not collinear) and $B_{i,n}(t) = \frac{n!}{i!(n-i)!}(1-t)^{n-i}t^i$ is the i-th Bézier polynomial of degree n. Denote by

$$\tau_1 = \frac{\mathbf{P}_3 - \mathbf{P}_0}{\|\mathbf{P}_3 - \mathbf{P}_0\|}, \qquad \tau_2 = \frac{\bar{\mathbf{P}} - \mathbf{P}_0 - \langle \bar{\mathbf{P}} - \mathbf{P}_0, \tau_1\rangle\tau_1}{\|\bar{\mathbf{P}} - \mathbf{P}_0 - \langle \bar{\mathbf{P}} - \mathbf{P}_0, \tau_1\rangle\tau_1\|}, \qquad \tau_3 = \tau_1 \times \tau_2. \tag{19}$$

Then $\mathbf{C}_\pi(t)$ has the presentation

$$\mathbf{C}_\pi(t) = \sum_{i=0}^{3}\tilde{\mathbf{P}}_iB_{i,3}(t), \tag{20}$$

where $\tilde{\mathbf{P}}_i = \mathbf{P}_i - \langle \mathbf{P}_i - \mathbf{P}_0, \tau_3\rangle\tau_3$.

Definition 0.1. $\mathbf{C}(t)$ is called a double-side curve if $\mathbf{C}_\pi(t)$ intersects to an inner point of the line segment $\mathbf{P}_0\mathbf{P}_3$; otherwise, it is called single-side.

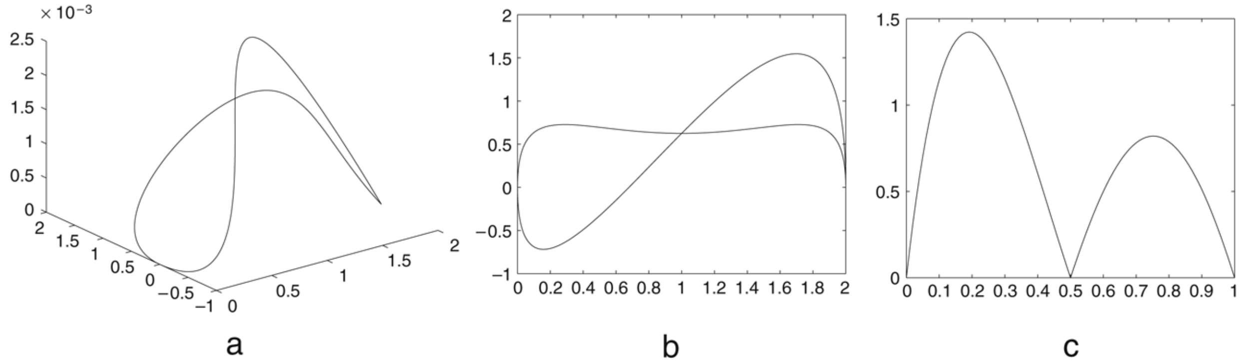

Fig. 1. (a) is the 3D figure of $\mathbf{C}(t)$ (the higher curve) and $\mathbf{F}(t)$, (b) is the projections of $\mathbf{C}(t)$ (the evener curve) and $\mathbf{F}(t)$ onto (x, y)-plane, and (c) is the figure of the error curve $c(t) = \|\mathbf{F}(t) - \mathbf{C}(t)\|$.

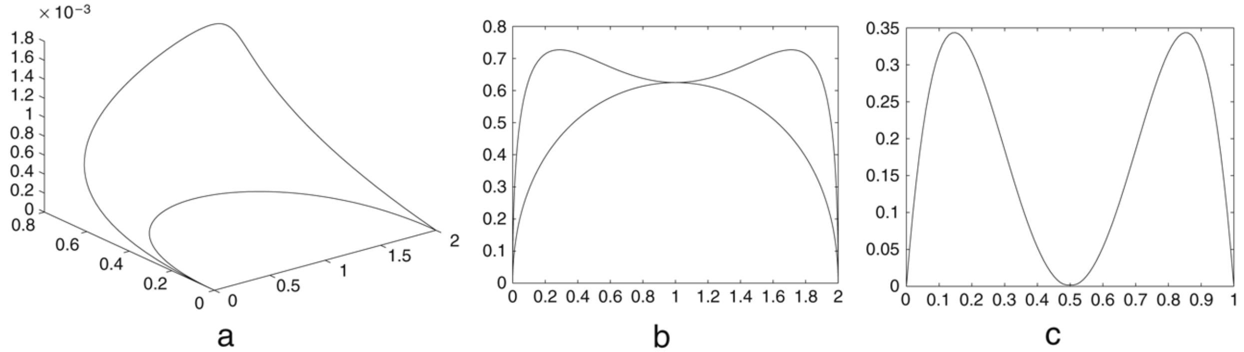

Fig. 2. (a) is the 3D figure of $\mathbf{D}(t)$ (the lower curve) and $\mathbf{F}(t)$, (b) is the projections of $\mathbf{D}(t)$ (the inner curve) and $\mathbf{F}(t)$ onto (x, y)-plane, and (c) is the figure of the error curve $d(t) = \|\mathbf{F}(t) - \mathbf{D}(t)\|$.

The curve $\mathbf{C}(t)$ showed in Fig. 1(a) is double-side and the curve $\mathbf{D}(t)$ showed in Fig. 2(a) is single-side. The following theorem tell us when $\mathbf{C}(t)$ is a double-side curve.

Theorem 0.1. *If $\mathbf{P}_3 - \mathbf{P}_0, \mathbf{T}_0, \mathbf{T}_3$ are not coplanar and the CIP has the solution $\mathbf{C}(t)$, then $\mathbf{C}(t)$ is a double-side curve if and only if*

$$\alpha\beta\langle \mathbf{T}_0, \tau_2\rangle\langle \mathbf{T}_3, \tau_2\rangle < 0. \tag{21}$$

Proof. Since τ_1, τ_2, τ_3 compose of an orthonormal coordinate system, the necessary and sufficient condition of $\mathbf{C}_\pi(t)$ being a point on the straight line determined by $\mathbf{P}_3$ and $\mathbf{P}_0$ is

$$\langle \mathbf{C}_\pi(t) - \mathbf{P}_0, \tau_2\rangle = 0. \tag{22}$$

According to (20), $\tilde{\mathbf{P}}_0 = \mathbf{P}_0$ and $\langle \tilde{\mathbf{P}}_3 - \mathbf{P}_0, \tau_2\rangle = \langle \mathbf{P}_3 - \mathbf{P}_0, \tau_2\rangle = 0$. Therefore, (22) becomes

$$t(1-t)(\langle \mathbf{P}_1 - \mathbf{P}_0, \tau_2\rangle(1-t) + \langle \mathbf{P}_2 - \mathbf{P}_0, \tau_2\rangle t) = 0. \tag{23}$$

Eq. (23) shows that the necessary and sufficient condition of $\mathbf{C}_\pi(t)$ intersecting the straight line segment $\mathbf{P}_0\mathbf{P}_3$ to an inner point is

$$\langle \mathbf{P}_1 - \mathbf{P}_0, \tau_2\rangle\langle \mathbf{P}_2 - \mathbf{P}_0, \tau_2\rangle < 0. \tag{24}$$

Since $\mathbf{P}_1 = \mathbf{P}_0 + \alpha\mathbf{T}_0$, $\mathbf{P}_2 = \mathbf{P}_3 + \beta\mathbf{T}_3$ and $\langle \mathbf{P}_2 - \mathbf{P}_0, \tau_2\rangle = \langle \mathbf{P}_2 - \mathbf{P}_3, \tau_2\rangle$, (24) is equivalent to (21). ♣

Corollary 0.1. *If $\mathbf{P}_3 - \mathbf{P}_0, \mathbf{T}_0, \mathbf{T}_3$ are not coplanar and the CIP has the solution $\mathbf{C}(t)$, then $\mathbf{C}(t)$ is a double-side curve if and only if*

$$\langle \tau_3, \mathbf{T}_0\rangle\langle \tau_3, \mathbf{T}_3\rangle\langle \tau_2, \mathbf{T}_0\rangle\langle \tau_2, \mathbf{T}_3\rangle > 0.$$

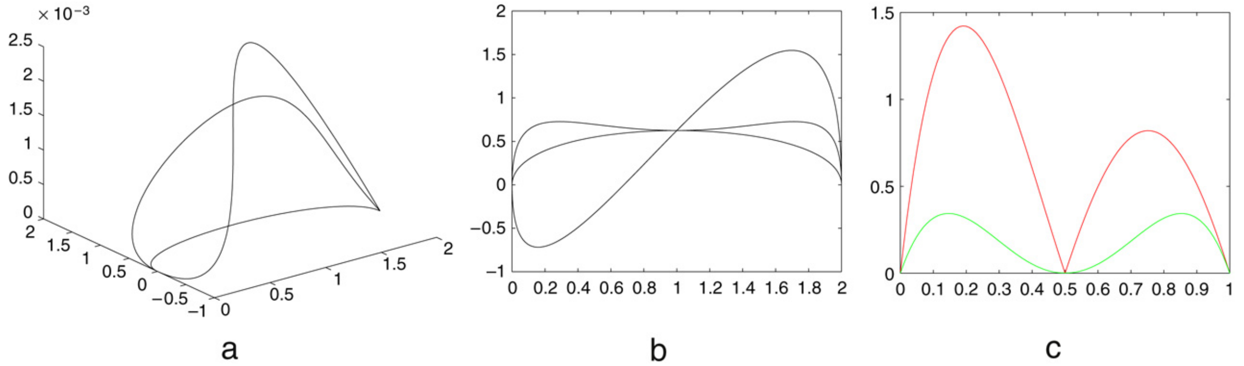

Fig. 3. (a) is the 3D figure of $\mathbf{C}(t)$ (red), $\mathbf{D}(t)$ (green) and $\mathbf{F}(t)$ (black), (b) is the projections of $\mathbf{C}(t)$ (red), $\mathbf{D}(t)$ (green) and $\mathbf{F}(t)$ (black) onto (x, y)-plane, and (c) is the figure of the error curves of $c(t) = \|\mathbf{F}(t) - \mathbf{C}(t)\|$ (red) and $d(t) = \|\mathbf{F}(t) - \mathbf{D}(t)\|$ (green). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. Since $\mathbf{P}_3 - \mathbf{P}_0, \mathbf{T}_0, \mathbf{T}_3$ are not coplanar, according to (8) both $\bar{\mathbf{P}} - \mathbf{P}_0, \mathbf{T}_0, \mathbf{T}_3$ and $\bar{\mathbf{P}} - \mathbf{P}_3, \mathbf{T}_0, \mathbf{T}_3$ are not coplanar. And

$$\begin{cases} \alpha = \dfrac{\left\langle (\mathbf{P}_3 - \mathbf{P}_0) \times \mathbf{T}_3, \bar{\mathbf{P}} - \mathbf{P}_0 \right\rangle}{3s^2 t \left\langle (\mathbf{P}_3 - \mathbf{P}_0) \times \mathbf{T}_3, \mathbf{T}_0 \right\rangle} = \dfrac{-\|(\bar{\mathbf{P}} - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0)\| \left\langle \tau_3, \mathbf{T}_3 \right\rangle}{3s^2 t \left\langle \mathbf{T}_3 \times \mathbf{T}_0, \mathbf{P}_3 - \mathbf{P}_0 \right\rangle}, \\ \beta = \dfrac{\left\langle (\mathbf{P}_3 - \mathbf{P}_0) \times \mathbf{T}_0, \bar{\mathbf{P}} - \mathbf{P}_0 \right\rangle}{3st^2 \left\langle (\mathbf{P}_3 - \mathbf{P}_0) \times \mathbf{T}_0, \mathbf{T}_3 \right\rangle} = \dfrac{-\|(\bar{\mathbf{P}} - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0)\| \left\langle \tau_3, \mathbf{T}_0 \right\rangle}{3s^2 t \left\langle \mathbf{T}_0 \times \mathbf{T}_3, \mathbf{P}_3 - \mathbf{P}_0 \right\rangle}. \end{cases} \tag{25}$$

Noting that $\mathbf{T}_3 \times \mathbf{T}_0 = -\mathbf{T}_0 \times \mathbf{T}_3$, $t > 0$ and $s = 1 - t > 0$, Corollary 0.1 is a direct conclusion of (25). ♣

Case 3: Two end tangents being nearly parallel. Corollary 0.1 shows that if $\mathbf{T}_3$ and $\mathbf{T}_0$ are nearly parallel and nearly orthogonal to τ_2 or τ_3, the perturbation of $\mathbf{T}_3$ or $\mathbf{T}_0$ may changes the shape of $\mathbf{C}(t)$ dramatically, i.e., from single-side curve to double-side curve or vice versa. This shows that in those cases one should avoid using methods that are sensitive to $\mathbf{T}_3$ and $\mathbf{T}_0$.

The following steps are used in [1] to construct the cubic Bézier curve passing through $\mathbf{P}_0, \bar{\mathbf{P}}, \mathbf{P}_3$ and tangenting $\mathbf{T}_0$ and $\mathbf{T}_3$ to $\mathbf{P}_0$ and $\mathbf{P}_3$, respectively.

1. Determine the plane Π by $\mathbf{P}_0, \mathbf{P}_3$ and $\mathbf{T}_0$;
2. Intersect Π with the line passing through $\bar{\mathbf{P}}$ and parallel to $\mathbf{T}_3$ to yield $\mathbf{P}_d$;
3. Obtain $\mathbf{P}_c$ by intersecting the line $\mathbf{P}_0\mathbf{P}_3$ with the line passing through $\mathbf{P}_d$ and parallel to $\mathbf{T}_0$.

It is clear that above method is sensitive to $\mathbf{T}_3$ and $\mathbf{T}_0$. In the following, we present three methods of not sensitive to $\mathbf{T}_3$ and $\mathbf{T}_0$. (8) shows that four vectors $\bar{\mathbf{P}} - \mathbf{P}_0, \mathbf{P}_3 - \mathbf{P}_0, \mathbf{T}_0$ and $\mathbf{T}_3$ should be nearly coplanar if $\mathbf{T}_0$ and $\mathbf{T}_3$ are nearly parallel. This means that it is reasonable to project these vectors onto a same plane, say Π.

We set $\mathbf{T}_0^{\text{new}} = \frac{\mathbf{T}'_0 + \langle \mathbf{T}'_0, \mathbf{T}'_3 \rangle \mathbf{T}'_3}{\|\mathbf{T}'_0 + \langle \mathbf{T}'_0, \mathbf{T}'_3 \rangle \mathbf{T}'_3\|}$ and $\mathbf{T}_3^{\text{new}} = \frac{\langle \mathbf{T}'_0, \mathbf{T}'_3 \rangle}{|\langle \mathbf{T}'_0, \mathbf{T}'_3 \rangle|} \mathbf{T}_0^{\text{new}}$, where $\mathbf{T}'_0$ and $\mathbf{T}'_3$ are the projections of $\mathbf{T}_0$ and $\mathbf{T}_3$, respectively. Then, we solve (13) to obtain the fitting curve. We give three methods to construct the plane Π. Our numerical results show that all of them produce fair curves fitting the original curves well. The first method is to require Π is the plane determined by three points $\bar{\mathbf{P}}, \mathbf{P}_0$ and $\mathbf{P}_3$. This method is to emphasize the interpolation of $\bar{\mathbf{P}}$. The second method is to fix the end tangents first, i.e., set $\mathbf{T}_0^{\text{new}} = \frac{\mathbf{T}_0 + \langle \mathbf{T}_0, \mathbf{T}_3 \rangle \mathbf{T}_3}{\|\mathbf{T}_0 + \langle \mathbf{T}_0, \mathbf{T}_3 \rangle \mathbf{T}_3\|}$ and $\mathbf{T}_3^{\text{new}} = \frac{\langle \mathbf{T}_0, \mathbf{T}_3 \rangle}{|\langle \mathbf{T}_0, \mathbf{T}_3 \rangle|} \mathbf{T}_0^{\text{new}}$ and require Π is the plane determined by $\mathbf{T}_0^{\text{new}}, \mathbf{P}_0$ and $\mathbf{P}_3$. This method is to emphasize the tangents of the two end points. The third method is to obtain $\mathbf{T}_0^{\text{new}}$ by solving the following optimization problem

$$\min\{\|\mathbf{T}_0^{\text{new}} \times \mathbf{T}_0\|^2 + \|\mathbf{T}_0^{\text{new}} \times \mathbf{T}_3\|^2 + \|\mathbf{T}_0^{\text{new}} \times \mathbf{T}'\|^2; \ \|\mathbf{T}_0^{\text{new}}\| = 1\},$$

where $\mathbf{T}' = \frac{(\bar{\mathbf{P}} - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0)}{(\bar{\mathbf{P}} - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0)}$. Similarly, we require Π is the plane determined by $\mathbf{T}_0^{\text{new}}, \mathbf{P}_0$ and $\mathbf{P}_3$.

The following example shows the results obtained by the methods of the book and this paper, respectively. Fig. 1. shows the results obtained by the method in the book. Fig. 2 shows the results obtained by our method. In Fig. 3 we put the results obtained by two methods together.

Let $\mathbf{F}(u) = \sum_{i=0}^{4} \mathbf{P}_{i,4} B_{i,4}(u)$ be the space curve to be approximated, where

$$\mathbf{P}_{0,4} = (0, 0, 0), \qquad \mathbf{P}_{1,4} = (0, 2, 0), \qquad \mathbf{P}_{2,4} = (1, -1, \sigma), \qquad \mathbf{P}_{3,4} = (2, 2, 2\delta), \qquad \mathbf{P}_{4,4} = (2, 0, 0)$$

with $\sigma = 0.004$, $\delta = 0.0005$. Thus, $\mathbf{T}_0 = (0, 1, 0)$, $\mathbf{P}_0 = (0, 0, 0)$, and $\mathbf{T}_3 = (0, -1, -\delta)/\sqrt{1 + \delta^2}$, $\mathbf{P}_3 = (2, 0, 0)$. We select $\bar{\mathbf{P}} = \mathbf{F}(0.5) = (1.0000, 0.6250, \frac{7}{4000})$. Using Matlab (or by direct calculation), we obtain that $\mathbf{P}_d \approx (1.000, -1.125, 0.000)$ and $\mathbf{P}_c \approx (1.000, 0.000, 0.000)$. Solving (9.102) ([1], p.445), i.e,

$$(1 - t)^3 + 3(1 - t)^2 t = \frac{\|\mathbf{P}_c - \mathbf{P}_3\|}{\|\mathbf{P}_0 - \mathbf{P}_3\|} \approx 0.500,$$

we obtain $t \approx 0.500$. Assume

$$\mathbf{P}_d - \mathbf{P}_c = a\mathbf{T}_0, \qquad \bar{\mathbf{P}} - \mathbf{P}_d = b\mathbf{T}_3,$$

we obtain

$$a = \langle \mathbf{P}_d - \mathbf{P}_c, \mathbf{T}_0 \rangle \approx -1.125, \qquad b = \langle \mathbf{P} - \mathbf{P}_d,\ \mathbf{T}_3 \rangle \approx -1.750. \tag{26}$$

Remark. Eq. (26) is not the same as (9.100) in the book (on page 444). (26) shows that it is not necessary that $a \geq 0$ or $b \leq 0$.

From $\alpha = \frac{a}{B_{1,3}(t)}$ and $\beta = \frac{b}{B_{2,3}(t)}$ ((9.99) in the book (on page 444)), we obtain

$$\alpha \approx -3.000, \qquad \beta \approx -4.667. \tag{27}$$

According to (14), it finally yields

$$\mathbf{C}(t) = s^3\mathbf{P}_0 + 3s^2t\mathbf{P}_1 + 3st^2\mathbf{P}_2 + t^3\mathbf{P}_3, \quad s = 1 - t, \tag{28}$$

where $\mathbf{P}_0 = (0, 0, 0)$, $\mathbf{P}_1 \approx (0, -3, 0)$, $\mathbf{P}_2 \approx (2, 4.667, 0.005)$, $\mathbf{P}_3 = (2, 0, 0)$.

Fig. 1 shows the curve $\mathbf{C}(t)$ determined by (28) does not fit $\mathbf{F}(t)$ well. Fig. 1(a) is the 3D figure of $\mathbf{C}(t)$ and $\mathbf{F}(t)$, (b) is the projections of $\mathbf{C}(t)$ and $\mathbf{F}(t)$ onto (x, y)-plane, and (c) is the figure of the error curve $c(t) = \|\mathbf{F}(t) - \mathbf{C}(t)\|$.

Noting that $\mathbf{T}_0 = (0, 1, 0)$ and $\mathbf{T}_3 = (0, -1, -\delta)/\sqrt{1+\delta^2}$, this example shows that the method in the book could fail in the case of two end tangents are nearly parallel.

The following curve is obtained by the second method presented in this paper.

$$\mathbf{D}(t) = s^3\mathbf{P}_0 + 3s^2t\mathbf{P}_1 + 3st^2\mathbf{P}_2 + t^3\mathbf{P}_3, \quad s = 1 - t,$$

where $\mathbf{P}_0 = (0, 0, 0)$, $\mathbf{P}_1 \approx (0, 0.8333, 0)$, $\mathbf{P}_2 \approx (2, 0.8333, 0)$, $\mathbf{P}_3 = (2, 0, 0)$. Fig. 2 is the corresponding figure to Fig. 1. Fig. 3 is the comparison of the original curve and the fitting curves. The results show that the fitting curve obtained by the method present in this paper is much better than the one obtained by the method given in [1].

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References

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